Introduction to Mathematical Quantum Theory

Solution to the Exercises

– 14.04.2020 –

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Exercise 1

Let V be a closed subspace of $\mathcal{H}$ Hilbert space. Let A be a linear bounded operator on $\mathcal{H}$ such that $A(V) \subseteq V$. Prove that $A^*\left(V^\perp\right) \subseteq V^\perp$.

Proof. Consider $\psi \in V^\perp$ and let $\varphi \in V$. We then get

$$\langle \varphi, A^*\psi \rangle = \langle A\varphi, \psi \rangle = 0,$$

because $A\varphi \in V$ and $\psi \in V^\perp$. Given that φ was generic, we get that $A^*\psi \in V^\perp$.

□

Exercise 2

Let $\mathcal{H}$ be an Hilbert space. Let A be a linear bounded operator on $\mathcal{H}$ with linear bounded inverse A^{-1}. Prove that $\left(A^{-1}\right)^* A^* = A^*\left(A^{-1}\right)^* = \mathrm{id}$. Deduce that A^* is invertible and that $(A^*)^{-1} = \left(A^{-1}\right)^*$.

Proof. Given that A is invertible we get that both A^* and $\left(A^{-1}\right)^*$ are well defined linear bounded operators. Recall that we proved before (see Exercise Sheet number 2) that $(AB)^* = B^*A^*$. We then get $\mathrm{id} = \mathrm{id}^* = \left(AA^{-1}\right)^* = \left(A^{-1}\right)^* A^*$. In a similar way, we also get $\mathrm{id} = \mathrm{id}^* = \left(A^{-1}A\right)^* = A^*\left(A^{-1}\right)^*$.

Now, given that $\left(A^{-1}\right)^* A^* = A^*\left(A^{-1}\right)^* = \mathrm{id}$ then A^* is invertible and $(A^*)^{-1} = \left(A^{-1}\right)^*$.

□

Exercise 3

Consider the Hilbert space $\mathcal{H} := \ell^2(\mathbb{N})$.

a Define the operator A as

$$(A\alpha)_n = \alpha_{n+1} \qquad \forall n \in \mathbb{N}, \tag{1}$$

for any $\alpha = \{\alpha_n\}_{n\in\mathbb{N}} \in \mathcal{H}$.

Prove that A is a well defined linear bounded operator, find its norm and its spectrum.

b Consider A^* the adjoint of A. Show its explicit action and find its norm and its spectrum.

c Define $B := A^*A$. Prove that B is a self-adjoint operator, show its explicit action and find its norm and its spectrum.

Hint: Recall that if T is a linear bounded operator, the spectrum $\sigma(T)$ is a closed set, $\rho(T) \equiv \mathbb{C} \setminus \sigma(T)$ the resolvent of T is defined as

$$\rho(T) := \left\{ \lambda \in \mathbb{C} | \ (T - \lambda \, \mathrm{id})^{-1} \text{ is a well-defined, linear, bounded operator} \right\}, \tag{2}$$

and that $\sigma(T) \subseteq \overline{B_{\|T\|}(0)}$, where $B_R(0) := \{\alpha \in \mathcal{H} | \ \|\alpha\|_2 < R\}$.

Proof. To prove **a**, first consider $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}, \ \beta = \{\beta_n\}_{n\in\mathbb{N}} \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. We get

$$(A(\alpha + \lambda\beta))_n = (\alpha + \lambda\beta)_{n+1} = \alpha_{n+1} + \lambda\beta_{n+1} = (A\alpha)_n + \lambda (A\beta)_n,$$

and therefore A is linear. To prove that is bounded consider $\alpha \in \mathcal{H}$; we get

$$\|A\alpha\|_2^2 = \sum_{n\geqslant 0} |(A\alpha)_n|^2 = \sum_{n\geqslant 1} |\alpha_n|^2 \leqslant \|\alpha\|_2^2,$$

therefore A is well defined from $\mathcal{H}$ to itself and $\|A\| \leqslant 1$. Let now $e^j = \{\delta_{j,n}\}_{n\in\mathbb{N}}$; on the one hand $\|e^j\|_2 = 1$, on the other we also get that for any $j > 0$ we get $\|Ae^j\|_2 = 1$, therefore $\|A\| = 1$.

Given that $\|A\| = 1$ we get that $\sigma(A) \subseteq \overline{B_1(0)}$. Consider now $\lambda \in B_1(0)$. If we look for a solution of $A\alpha = \lambda\alpha$, we get that such α needs to satisfy

$$\alpha_{n+1} = \lambda\alpha_n.$$

It is easy to see that $\alpha_n := \lambda^n\alpha_0$ satisfies the equation, and given that

$$\|\alpha\|_2^2 = \sum_{n\geqslant 0} |\lambda|^{2n} |\alpha_0|^2 = \frac{|\alpha_0|^2}{1 - |\lambda|^2}$$

we also get that $\alpha \in \mathcal{H}$. This implies that α is an eigenvector for A and as a consequence $B_1(0) \subseteq \sigma(A)$. Given that the spectrum is always a closed set we get $\overline{B_1(0)} \subseteq \overline{\sigma(A)} = \sigma(A) \subseteq \overline{B_1(0)}$, and hence $\sigma(A) = \overline{B_1(0)}$.

To prove **b**, let $\alpha, \ \gamma \in \mathcal{H}$. We get

$$\sum_{n\geqslant 0} \overline{\gamma_n} (A^*\alpha)_n = \langle \gamma, A^*\alpha \rangle = \langle A\gamma, \alpha \rangle = \sum_{n\geqslant 0} \overline{(A\gamma)_n}\alpha_n = \sum_{n\geqslant 0} \overline{\gamma_{n+1}}\alpha_n = \sum_{n\geqslant 1} \overline{\gamma_n}\alpha_{n-1}.$$

Given that α and γ were arbitrary we get that

$$(A^*\alpha)_n := (1 - \delta_{n,0})\,\alpha_{n-1} \equiv \begin{cases} \alpha_{n-1} & \text{if } n > 0, \\ 0 & \text{if } n = 0. \end{cases}$$

From the definition we easily get that $\|A^*\alpha\|_2 = \|\alpha\|_2$, and therefore $\|A^*\| = 1$.

If we now turn to the spectrum, we get that given that $\|A^*\| = 1$, then $\sigma(A^*) \subseteq \overline{B_1(0)}$. Consider now $\lambda \in B_1(0)$ and let $\gamma \in \mathcal{H}$. We look for α so that $(A^* - \lambda\,\mathrm{id})\,\alpha = \gamma$. Then we have

$$\begin{aligned} \alpha_{n-1} - \lambda\alpha_n &= \gamma_n, & \text{if } n > 0, \\ -\lambda\alpha_0 &= \gamma_0. \end{aligned}$$

As a consequence, we can sum up the coefficients to get

$$\begin{aligned} \sum_{j=1}^{n} \lambda^j \left(\alpha_j - \frac{1}{\lambda}\alpha_{j-1}\right) &= \sum_{j=1}^{n} \lambda^j \alpha_j - \sum_{j=1}^{n} \lambda^{j-1}\alpha_{j-1} \\ &= \sum_{j=1}^{n} \lambda^j \alpha_j - \sum_{j=0}^{n-1} \lambda^j \alpha_j = \lambda^n \alpha_n - \alpha_0. \end{aligned}$$

On the other hand, we use the fact that $(A^* - \lambda\,\mathrm{id})\,\alpha = \gamma$ to get

$$\alpha_n = \lambda^{-n}\left(\sum_{j=1}^{n} \lambda^j \left(\alpha_j - \frac{1}{\lambda}\alpha_{j-1}\right) + \alpha_0\right) = -\lambda^{-(n+1)} \sum_{j=0}^{n} \lambda^j \gamma_j.$$

If $|\lambda| < 1$, it is easy to see that there exist $\gamma \in \mathcal{H}$ so that $|\alpha_n| \to +\infty$ as $n \to +\infty$, and therefore $A^* - \lambda\,\mathrm{id}$ does not have an inverse from $\mathcal{H}$ to itself. As a consequence $B_1(0) \subseteq \sigma(A^*)$, and given that the spectrum is closed, we get $\overline{B_1(0)} \subseteq \sigma(A^*) \subseteq \overline{B_1(0)}$, which implies $\sigma(A^*) = \overline{B_1(0)}$.

To prove **c**, a simple computation first gives that $(B\alpha)_n = (1 - \delta_{n,0})\,\alpha_n$. From this it is easy to see that $\|B\| = 1$. B is also self-adjoint because we get $B^* = (A^*A)^* = A^*A^{**} = B$. Given that $Be^0 = 0$ and that $Be^j = e^j$ for any $j > 0$, we also get that $\{0, 1\} \subseteq B$. Given that B is self-adjoint, $\sigma(B) \subseteq \mathbb{R}$. Let now $\lambda \in \mathbb{R}\backslash\{0, 1\}$. If we consider the equation $(B - \lambda\,\mathrm{id})\,\alpha = \gamma$, we get that fixed $\gamma \in \mathcal{H}$, α needs to be

$$\begin{aligned} (1 - \lambda)\,\alpha_n &= \gamma_n, & \text{if } n > 0, \\ -\lambda\alpha_0 &= \gamma_0, \end{aligned}$$

and as a consequence we can define

$$\left((A - \lambda\,\mathrm{id})^{-1}\gamma\right)_n := \begin{cases} \frac{1}{1-\lambda}\gamma_n & \text{if } n > 0, \\ -\frac{1}{\lambda}\gamma_0 & \text{if } n = 0, \end{cases}$$

and this is a well defined linear bounded operator, implying that $\lambda \in \rho(B)$. We then conclude that $\sigma(B) = \{0, 1\}$.

□

Exercise 4

Consider the interval $I = (a, b) \subseteq \mathbb{R}$ and the Hilbert space $\mathcal{H} := L^2(I)$. Consider $\varphi \in C(I)$ a real valued continuous function with $\|\varphi\|_\infty < +\infty$. Consider the operator T_φ defined for any $\psi \in \mathcal{H}$ as

$$T_\varphi \psi(x) := \varphi(x)\,\psi(x). \tag{3}$$

Prove that T_φ is a well defined linear bounded operator and prove that $\sigma(T_\varphi) = \overline{\varphi(I)}$.

Hint: Show first that $\varphi(I) \subseteq \sigma(T_\varphi)$ and use the fact that the spectrum is closed to show that the same is true for the closures. Next, show that $\left(\overline{\sigma(T_\varphi)}\right)^{\mathrm{c}} \subseteq \rho(T_\varphi)$ to conclude.

Proof. Let $y_0 \in \varphi(I)$ and let $x_0 \in I$ such that $\varphi(x_0) = y_0$. Consider the sequence given by

$$\psi_n(x) := \begin{cases} \sqrt{n} & |x - x_0| \leqslant \frac{1}{2n}, \\ 0 & |x - x_0| > \frac{1}{2n}. \end{cases} \tag{4}$$

We then get that $\|\psi_n\|_2 = 1$ and therefore

$$\begin{aligned} \lim_{n\to+\infty} \frac{\|T_\varphi \psi_n - y\psi_n\|_2}{\|\psi_n\|_2} &= \lim_{n\to+\infty} \left(\sqrt{n} \int_{x_0-\frac{1}{2n}}^{x_0+\frac{1}{2n}} (\varphi(x) - y)\, dx\right)^{\frac{1}{2}} \\ &= \lim_{n\to+\infty} \frac{1}{\sqrt[4]{n}} \left(n \int_{x_0-\frac{1}{2n}}^{x_0+\frac{1}{2n}} \varphi(x)\, dx - y\right)^{\frac{1}{2}}. \end{aligned}$$

From the mean value theorem for integrals, given that φ is a continuous function, we get that

$$\lim_{n\to+\infty} n \int_{x_0-\frac{1}{2n}}^{x_0+\frac{1}{2n}} \varphi(x)\, dx = \varphi(x_0) = y,$$

and as a consequence

$$\lim_{n\to+\infty} \frac{\|T_\varphi \psi_n - y\psi_n\|_2}{\|\psi_n\|_2} = 0.$$

As we saw in class, this implies that $y \in \sigma(T_\varphi)$; this implies that $\varphi(I) \subseteq \sigma(T_\varphi)$, and given that the spectrum is closed we get that $\overline{\varphi(I)} \subseteq \sigma(T_\varphi)$.

On the other hand, let $\lambda \notin \varphi(I)$; then the operator $(T - \lambda\,\mathrm{id})^{-1}$ is defined as

$$(T - \lambda\,\mathrm{id})^{-1} \psi(x) = \frac{1}{\varphi(x) - \lambda} \psi(x),$$

and its norm is bounded by $\left\|(T - \lambda\,\mathrm{id})^{-1}\right\| \leqslant \sup_{x\in\mathbb{R}} |\varphi(x) - \lambda|^{-1}$, which is finite by hypotheses. As a consequence we get that $\sigma(T_\varphi) = \overline{\varphi(I)}$.

□